# SINGULARITIES, TEST CONFIGURATIONS AND CONSTANT SCALAR CURVATURE KÄHLER METRICS

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Abstract. In this paper we extend the notion of Futaki invariant to big and nef classes in such a way that it defines a continuous function on the Kähler cone up to the boundary. We apply this concept to prove that reduced normal crossing singularities are sufficient to check $K$-instability of a smooth polarized manifold. Similar improvement on Donaldson's lower bound for Calabi energy is given. The effect of resolving singularities of the central fiber of a given test configuration is also studied, providing new examples of manifolds which do not admit Kähler constant scalar curvature metrics in some classes.

## 1. Introduction

One the most fascinating problems in complex differential geometry is certainly the existence problem for Kähler metrics of constant scalar curvature in a fixed integral cohomology class (Einstein metrics are an important example). While, at least in the first place, one is primarly interested in studying such a problem on a smooth manifold, singular spaces almost immediately enter the scene at least for two important reasons.

On the one hand, when trying to construct them with what is probably at the moment the tool with the highest chance of success, namely via Ricci or Calabi flows, with different specific difficulties, one faces the questions of whether and how these flows develop singularities. On the other hand, the most important obstruction to the existence of such metrics is a partial answer to the Tian-Yau-Donaldson Conjecture ([24], [25], [26], [8]) by which we know that the existence of special metrics implies a suitably adapted GIT stability notion of the corresponding algebraic *polarized* manifold. Hence one naturally asks which type of singularities must be introduced to make the least effort to destabilize a smooth manifold without Kcsc metrics. This is the central problem studied in this paper.

What is now believed to be the right stability notion entering into this picture is the so called *K-stability* introduced by Tian ([24], [25]) (but essentially going back to Ding-Tian [6]) and Donaldson ([8]), building on previous work by Futaki and Calabi. The mutual

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relationship between these, and other notions has been deeply investigated by Paul-Tian in [18] and some of their results will be recalled and used in our work. Let us now recall Donaldson's definition.

**Definition 1.1.** (1) Let $(V, L)$ be a $n$-dimensional polarized variety or scheme. Given a one parameter subgroup $\rho : \mathbb{C}^* \to \mathrm{Aut}(V)$ with a linearization on $L$ and denoted by $w(V, L)$ the weight of the $\mathbb{C}^*$-action induced on $\bigwedge^{\mathrm{top}} H^0(V, L)$, we have the following asymptotic expansions as $k \gg 0$:

$$h^0(V, L^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}) \tag{1}$$

$$w(V, L^k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}) \tag{2}$$

The (normalized) *Futaki invariant* of the action is

$$F(V, L, \rho) = \frac{b_1}{a_0} - \frac{b_0\, a_1}{a_0^2}.$$

(2) A *test configuration* $(X, L) \to \mathbb{C}$ of a polarized manifold $(M, A)$ consists of a scheme $X$ endowed with a $\mathbb{C}^*$-action that linearizes on a line bundle $L$ over $X$, and a flat $\mathbb{C}^*$-equivariant map $f : X \to \mathbb{C}$ such that $L|_{f^{-1}(0)}$ is ample on $f^{-1}(0)$ and we have $(f^{-1}(1), L|_{f^{-1}(1)}) \simeq (M, A^r)$ for some $r > 0$.
When $(M, A)$ has a $\mathbb{C}^*$-action $\rho : \mathbb{C}^* \to \mathrm{Aut}(M)$, a test configuration where $X = M \times \mathbb{C}$ and $\mathbb{C}^*$ acts on $X$ diagonally trought $\rho$ is called *product configuration.*

(3) The polarized manifold $(M, A)$ is *$K$-stable* if for each test configuration for $(M, A)$ the Futaki invariant of the induced action on the central fiber $(f^{-1}(0), L|_{f^{-1}(0)})$ is less than or equal to zero, with equality if and only if we have a product configuration.
A test configuration $(X, L) \to \mathbb{C}$ is called *destabilizing* if the Futaki invariant of the induced action on $(f^{-1}(0), L|_{f^{-1}(0)})$ is greater than zero.

Even though $(X, L)$ is indeed a smooth manifold, a destabilizing test configuration will then be just a scheme as well as its central fiber, while on the other hand it is easy to prove that the total space of the any test configuration with smooth central fiber is itself smooth. The first natural thing to do when singularities appear is then just to perform a resolution of singularities of the central fiber. This seemingly innocent operation has the horrible effect of changing both the generic fiber and its polarization, becoming then useless for the purpose of drawing conclusions about existence or nonexistence of special metrics on the initial manifold. We perform this operation on a simple example due to Ding-Tian, where the central fiber has only one isolated singular point, which can then be resolved by a double blow up.
The interest in this example is not just as a warning against the naive idea of resolving the singularities on the central fiber. Indeed, even paying the price of this change in manifold

and polarization, we prove that we can control the sign of the Futaki invariant on the new manifold at least for those Kähler classes which are sufficiently close to the pullback, via the resolution of singularities, of the old polarization on the initial manifold. This, en passant, gives a method of producing many new manifolds whithout Kcsc metrics in some classes, which is clearly of independent interest.
The main technical point is that when performing a resolution of singularities, the pullback of ample line bundles are just big and nef. This raises the natural question of whether it is possible to define a generalized Futaki invariant also for these (non Kähler) classes so as to have good continuity properties of this Futaki invariant when moving from the interior of the Kähler cone to these boundary points.
This is accomplished in Section 2 where this new definition is given and its continuity properties are proved (Proposition 2.4 and Theorem 2.6) under the assumption that the base space is a smooth algebraic manifold.
Desingularizing and the smoothing the central fiber is therefore not the right thing to do. On the other hand one can reduce the singularities of the whole test configurationiand perform some sort of embedded resolution of singularitites of the central fiber, so as to leave the generic fiber and its polarization untouched. Of course there is a variety of ways for doing this and in general there would be no hope of keeping control of the associated Futaki invariants, or even to preserve the test configuration structure.
The main result of this paper follows from choosing among all possibilities Mumford's semi-stable reduction theorem in its equivariant form (as it essentially follows from an application of a remark of Kollár's on equivariant resolution of singularities [15]). This procedure is essentially made of two steps, a *base change*, i.e. a suitable riparametrization of the base $\mathbb{C}$, and a *blow up* of an ideal sheaf supported over the central fiber. For both these operations we can control the generalized Futaki invariant, following a substantial extension of the above mentioned construction in the smooth case, to the case of general schemes. This requires passing through a much more abstract and deep machinery, already used by many authors such as Ross-Thomas [19], Paul-Tian[18], Fine-Ross [10]. We believe all these extensions of known concepts to the case of big and nef are of independent interest, yet, focusing on the main question addressed in this paper, all these results bring us to the main result:

**Theorem 1.2.** *Let $(X, L) \to \mathbb{C}$ be a destabilizing test configuration for a smooth polarised manifold, then there is a destabilizing test configuration $(X', L') \to \mathbb{C}$ for the same polarized manifold with smooth $X'$ and whose central fibre is a reduced simple normal crossing divisor.*

The proof of this result relies on a previous analysis by Ross-Thomas ([20]) on the behaviour of the Futaki invariant under blow up of subschemes supported on the central fiber with the pullback line bundle which would give just a big and nef line bundle and

not a genuine polarization. The new ingredients of our proof are then the mentioned continuity of the Futaki invariant under deformations of the polarization near the boundary of the Kähler cone, and Mumford's semi-stable reduction theorem and its equivariant incarnation.

The moral is then only mild singularities are necessary to destabilize a polarized manifold. The fact that only reduced simply normal crossing central fibers are relevant to detect the $K$-stability of a manifold strongly suggest to look for an analytic definition of the Futaki invariant for reduced simple normal crossing varieties which is at present missing (Ding-Tian's defintion requires normality of the variety) which should share the same continuity properties of the algebraic definition that we provide in this paper. We leave this analtyic problem to future investigation.

While in general we believe our result is optimal, we should also note that in case of the anti-canonical polarization (i.e. when looking for Einstein metrics) Tian has pointed out deep reasons culminating in Conjecture 1.4 in [25] to believe that normal reduced central fibers should be enough to destabilize an unstable manifold. We plan to attack this problem in the near future.

Of course one would be tempted to interpret Theorem 1.2 in terms of possible singularities developed by parabolic flows which tend to construct constant scalar curvature Kähler metrics. In order to put this intuition on solid grounds one should definitely look at the Calabi flow, i.e. the gradient flow of the K-energy (whose derivative is given by the Futaki invariant). Nevertheless many important pieces of informations are still missing to turn this evidence into a real Theorem. In particular, while we know that the Calabi flow is indeed asymptotic to a geodesic of Kähler potentials ([1]), there is still no method to associate a test configuration to a geodesic, leaving still plenty of room for wild speculation.

Staying on the algebraic side of this story, one is certainly interested in finding and characterizing the singularities of the (if any!) test configurations with the worst (i.e. bigger) Futaki invariant. The interest in such a problem comes form a result by Donaldson ([9]) which states that the Calabi functional is bounded below by the supremum of the Futaki invariants over all possible test configurations, suitably normalized so to make vacuous the trivial possible rescalings of a fixed test configuration given. Whether such a supremum is indeed obtained is studied by Székelyhidi in [23]. Following this circle of ideas we can prove two results:

**Theorem 1.3.** *Let $(X, L) \to \mathbb{C}$ be a test configuration for a smooth polarized manifold and assume that $X_{\mathrm{non-normal}}$ has codimension one. Then there exists a test configuration $(X', L') \to \mathbb{C}$ for the same polarized manifold with smooth $X'$, whose central fibre is a reduced simple normal crossing divisor, and*

$$F(X'_0, L'_0) > d\, F(X_0, L_0)$$

*for some integer $d > 0$.*

When looking for the "optimal" test configuration, i.e. the most destabilizing one, one is forced to introduce some normalization in order to avoid the possibility of enlarging arbitrarily the Futaki invariant for example by covering of the base. Futaki-Mabuchi [11], Székelyhidi [22] and Donaldson in [9] have proposed a natural normalization which will be recalled in Section 5. Our results then imply that an optimal test configuration in this sense has only reduced simple normal crossing central fibers, or its normalized Futaki invariant can be arbitrarily approximated by the one of test configurations with only reduced simple normal crossing central fibers.

**Theorem 1.4.** *Let $\Psi(X, L)$ be Donaldson's normalized Futaki invariant ([9]). Then*

$$\sup\{\Psi(X,L) \mid (X,L) \text{ is a test configuration of } (M,A)\} =$$
$$= \sup\{\Psi(X',L') \mid (X',L') \text{ is smooth with reduced simple normal crossing central fiber}\}.$$

The above result and the mentioned Donaldson's theorem then imply that also for the quest of the lower bound of the Calabi energy we can restrict ourselves to test configurations with mild singularities. Donaldson ([9]) conjectures that the number $sup\{\Psi(X,L) \mid (X,L) \text{ is a test configuration}\}$ gives in fact the exact lower bound for the Calabi energy in a fixed cohomology class, and this has been verified by Székelyhidi ([23]) for toric varities.
As we tried to explain in this introduction, all the results of this paper are interpreted in differential geometric terms. On the other hand the algebraic machinery to prove these results is quite heavy and at points very heavy. In particular Knudsen-Mumford theory is necessary and certainly not common place for differential geometers. We have hence decided to insert a brief appendix on this approach leading to Theorem 6.35 and its proof, which is a crucial result for our paper and while not original should be now accessible to non experts.

## 2. Donaldson-Futaki invariant of linearized line bundles

As mentioned in the introduction, the understanding of the role of singularities for K-stability needs a "good" extension of the Donaldson-Futaki invariant [8] to the boundary of the ample cone and in particular to nef and big line bundles in such a way to have a continuity when approaching the boundary. The aim of this section is to show that the following defintion achives this goal at least for smooth manifolds, postoponing the singular case to Section 4:

**Definition 2.1.** Let $V$ be a projective variety or scheme endowed with a $\mathbb{C}^*$-action and let $L$ be a big and nef line bundle on $V$. Choosing a linearization of the action on $L$ gives a $\mathbb{C}^*$-representation on $\bigoplus_{j=0}^{\dim V} H^j(V, L^k)^{(-1)^j}$. We set $w(V, L^k) = \operatorname{tr} A_k$, where $A_k$

is the generator of that representation. As $k \to +\infty$ we have the following expansion (see Corollary 6.23)

$$\frac{w(V,L^k)}{\chi(V,L^k)} = F_0 k + F_1 + O(k^{-1}),$$

and we define

$$F(V,L) = F_1$$

to be the *Donaldson-Futaki invariant* of the chosen action on $(V,L)$.

*Remark* 2.2. Clearly $w(V,L^k)$ is nothing but the weight of the induced $\mathbb{C}^*$-action on the Knudsen-Mumford characteristic $\mu(V,L^k) = \bigotimes_{j=0}^{\dim V} \det H^j(V,L^k)^{(-1)^j}$.

*Remark* 2.3. As is well known, when $L$ is ample $H^q(V,L^k) = 0$ for $q \geq 1$, thus we recover Donaldson's definition of the Futaki invariant. In particular $F(V,L)$ is calculated by means of the induced actions on the spaces of sections $H^0(V,L^k)$ for $k \gg 0$. On the other hand, this is not possible in general if $L$ is only nef and big. In this case, setting $n = \dim V$, as $k \to +\infty$ we have $h^0(V,L^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$ with $a_0 > 0$ by bigness and $h^q(V,L^k) = O(k^{n-q})$ for $q > 0$ by nefness, so that $h^1(V,L^k) = a_1' k^{n-1} + O(k^{n-2})$ with $a_1' \geq 0$. Analogously the weights of the induced actions on $H^0(V,L^k)$ and $H^1(V,L^k)$ are given respectively by $b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$ and $b_1' k^n + O(k^{n-1})$, thus by definition 2.1 we have

$$F(V,L^k) = \frac{a_0(b_1 - b_1') - (a_1 - a_1')b_0}{a_0^2} = \frac{a_0 b_1 - a_1 b_0}{a_0^2} - \frac{a_0 b_1' - a_1' b_0}{a_0^2}.$$

Hence for a nef and big line bundle $L$ on $V$, the Donaldson-Futaki invariant can be computed by means of the induced actions on $H^0(V,L^k)$ under the additional hypothesis $h^1(V,L^k) = O(k^{n-2})$. This represents the crucial technical difference in the definition of the Futaki invariant for big and nef line bundles of this paper and the one given by Ross and Thomas which takes into account just the contribution given by $H^0(V,L)$. The next Proposition shows why ours should be the right one for geometrical applications.

The fundamental continuity property we'll need can be stated in the following form:

**Proposition 2.4.** *Let $A$, $L$ be respectively an ample and a big and nef line bundle on a smooth projective manifold $M$. We have*

$$F(M, L^r \otimes A) = F(M,L) + O\left(\frac{1}{r}\right).$$

*Proof.* Fix an hermitian metrics on $L$ that is invariant with respect to the action of $S^1 \subset \mathbb{C}^*$ and suppose that the curvature $\omega$ is a Kähler metric. Since $L$ is nef, for each $r > 0$ we can choose an invariant metric on $L$ whose curvature $\eta_r$ satisfy $r\eta_r + \omega > 0$. In other words $r\eta_r + \omega$ is a Kähler form which coincides with the curvature of the induced

hermitian metric on the line bundle $L^r \otimes A$. Setting $n = \dim(M)$, by Riemann-Roch for each $r > 0$ we have:

$$\chi(M, L^{rk} \otimes A^k) = \left(\frac{k}{r}\right)^n \int_M \frac{(\eta_r + \frac{1}{r}\omega)^n}{n!} + \left(\frac{k}{r}\right)^{n-1} \int_M \frac{(\eta_r + \frac{1}{r}\omega)^{n-1} \wedge \mathrm{Ric}(\eta_r + \frac{1}{r}\omega)}{2(n-1)!} + O(k^{n-2}).$$

Now let $v$ be the holomorphic vector field on $M$ generating the given $\mathbb{C}^*$-action. Let $f$ and $g_r$ be smooth $S^1$-invariant functions on $M$ such that $i_v\omega + \bar{\partial} f = 0$ and $i_v\eta_r + \bar{\partial} g_r = 0$. Applying, in a similar fashion, the equivariant Riemann-Roch theorem, the weight of the induced action on $\bigotimes_{j=0}^n \det H^j(M, L^{rk} \otimes A^k)^{(-1)^j}$ is calculated as:

$$w(M, L^{rk} \otimes A^k) = \left(\frac{k}{r}\right)^{(n+1)} \int_M (g_r + \frac{1}{r} f) \frac{(\eta_r + \frac{1}{r}\omega)^n}{n!} + \left(\frac{k}{r}\right)^n \int_M (g_r + \frac{1}{r} f) \frac{(\eta_r + \frac{1}{r}\omega)^{n-1} \wedge \mathrm{Ric}(\eta_r + \frac{1}{r}\omega)}{2(n-1)!} + O(k^{n-1}).$$

As $r \to +\infty$ we have expansions:

$$\begin{aligned} \int_M \frac{(\eta_r + \frac{1}{r}\omega)^n}{n!} &= \int_M \frac{\eta_r^n}{n!} + O\left(\frac{1}{r}\right) = \frac{c_1(L)^n}{n!} + O\left(\frac{1}{r}\right) \\ \int_M \frac{(\eta_r + \frac{1}{r}\omega)^{n-1} \wedge \mathrm{Ric}(\eta_r + \frac{1}{r}\omega)}{2(n-1)!} &= \int_M \frac{\eta_r^{n-1} \wedge \mathrm{Ric}(\eta_r + \frac{1}{r}\omega)}{2(n-1)!} + O\left(\frac{1}{r}\right) \\ &= \frac{c_1(L)^{n-1} c_1(M)}{2(n-1)!} + O\left(\frac{1}{r}\right) \end{aligned}$$

$$\begin{aligned} \int_M (g_r + \frac{1}{r} f) \frac{(\eta_r + \frac{1}{r}\omega)^n}{n!} &= \int_M g_r \frac{\eta_r^n}{n!} + O\left(\frac{1}{r}\right) = \frac{c_1^T(L)^{n+1}}{(n+1)!} + O\left(\frac{1}{r}\right) \\ \int_M (g_r + \frac{1}{r} f) \frac{(\eta_r + \frac{1}{r}\omega)^{n-1} \wedge \mathrm{Ric}(\eta_r + \frac{1}{r}\omega)}{2(n-1)!} &= \int_M g_r \frac{\eta_r^{n-1} \wedge \mathrm{Ric}(\eta_r + \frac{1}{r}\omega)}{2(n-1)!} + O\left(\frac{1}{r}\right) \\ &= \frac{c_1^T(L)^n c_1^T(M)}{2\,n!} + O\left(\frac{1}{r}\right), \end{aligned}$$

where $c_1^T$ denotes the equivariant first Chern class. Thus we have:

$$F(M, L^r \otimes A) = \frac{\frac{c_1^T(L)^n c_1^T(M)}{2n!} \cdot \frac{c_1(L)^n}{n!} - \frac{c_1^T(L)^{n+1}}{(n+1)!} \cdot \frac{c_1(L)^{n-1} c_1(M)}{2(n-1)!}}{\left(\frac{c_1(L)^n}{n!}\right)^2} + O\left(\frac{1}{r}\right).$$

and the thesis follows. □

On a normal projective manifold, Paul-Tian ([18]) have proved the equivalence between Donaldson and Ding-Tian definitions of Futaki invariant. Since we need this result in the next section, and we found a simple and self-contained proof, we include it in the following:

**Proposition 2.5.** *Let $(V, L)$ be a polarized normal projective variety endowed with a $\mathbb{C}^*$-action. Let $\pi : \tilde{V} \to V$ be an equivariant resolution of singularities. We have*

$$F(V, L) = F(\tilde{V}, \pi^* L).$$

*Proof.* By Zariski's Main theorem we have an equivariant isomorphism $\pi_* \mathcal{O}_{\tilde{V}} \simeq \mathcal{O}_V$, thus the projection formula yields:

$$\chi(\tilde{V}, \pi^* L^k) = \chi(V, \pi_* \mathcal{O}_{\tilde{V}} \otimes L^k) = \chi(V, L^k),$$

and by the same token:

$$w(\tilde{V}, \pi^* L^k) = w(V, L^k).$$

The thesis follows by definition 2.1. □

Combining propositions 2.4 and 2.5 we get the following result, which explains the behaviour for normal varieties, of the Futaki invariant under Hironaka's resolution of singularities process:

**Theorem 2.6.** *Let $(V, L)$ be a polarized normal variety endowed with a $\mathbb{C}^*$-action. Let $\pi : \tilde{V} \to V$ be an equivariant resolution of singularities with exceptional divisor $E$. Then we have*

$$F\left(\tilde{V}, \pi^* L^r \otimes \mathcal{O}\left(-E\right)\right) = F(V, L) + O\left(\frac{1}{r}\right) \qquad \text{as } r \to +\infty.$$

*In particular, if $F(V, L) \neq 0$, then $\tilde{V}$ admits no cscK metrics in the classes $r\pi^* c_1(L) - E$ for $r \gg 0$.*

*Proof.* Since $\pi$ is a sequence of blow-ups along smooth centers, $\pi^* L$ is big and nef, moreover there exists $r_0 > 0$ such that $\pi^* L^{r_0} \otimes \mathcal{O}(-E)$ is ample. Thus, as $r \to +\infty$ we obtain:

$$\begin{aligned} F\left(\tilde{V}, \pi^* L^r \otimes \mathcal{O}\left(-E\right)\right) &= F\left(\tilde{V}, \pi^* L^{r-r_0} \otimes \pi^* L^{r_0} \otimes \mathcal{O}\left(-E\right)\right) \\ &= F\left(\tilde{V}, \pi^* L^{r-r_0}\right) + O\left(\frac{1}{r}\right) \\ &= F(V, L) + O\left(\frac{1}{r}\right), \end{aligned}$$

where the second and third equalities follow from propositions 2.4 and 2.5 respectively. □

## 3. Resolution of singularities and Ding-Tian example

Before digging into the main technical difficulties of the paper, let us show on an explicit example the bright, and especially the dark sides of the use of Theorem 2.6 on a test configuration. In fact we will construct $K$-destabilizing degenerations with smooth central fiber of a (non Fano, see [5, Theorem 1]) smooth surface obtained by resolution of singularities of a mild singular instable cubic surface found by Ding-Tian ([6]). As explained in the introduction, overcoming the unwanted features of the application of Theorem 2.6 is the core of the next sections.
Following Ding-Tian, let us the look at $X_f \subset \mathbb{P}^3$ be the zero locus of $f = z_0 z_1^2 + z_2 z_3 (z_2 - z_3)$.
Let us collect some elementary facts that the reader can easily verify with standard machinery:

(1) $X_f$ is singular only at $p_0 = (1:0:0:0)$;
(2) Having set $(x, y, z)$ affine co-ordinates centered on $(1:0:0:0)$, $\tilde{\mathbb{C}}^3 = \{((x, y, z), (l_0 : l_1 : l_2)) \in \mathbb{C}^3 \times \mathbb{P}^2 \,|\, x l_1 - y l_0 = x l_2 - z l_0 = y l_2 - z l_1 = 0\}$ be the blow-up of $\mathbb{C}^3$ at the origin with exceptional divisor $E \simeq \mathbb{P}^2$ and $\tilde{X}_f$ the proper transform of $X_f$, then $\tilde{X}_f$ is singular at points $(0:1:0), (0:0:1), (0:1:1) \in E$.
(3) Having set $\tilde{\tilde{X}}_f$ the proper transform of $\tilde{X}_f$ under the blow-up of $\tilde{\mathbb{C}}^3$ at points $(0:1:0), (0:0:1), (0:1:1) \in E$ and $E_1$ the exceptional divisor over $(0:1:0)$, then $\tilde{\tilde{X}}_f$ is smooth around $E_1$. Analogously, if $E_2$ is the exceptional divisor over $(0:0:1)$ and $E_3$ the one over $(0:1:1)$, then $\tilde{\tilde{X}}_f$ is smooth around $E_2$ and $E_3$ too. For future reference let $E_0$ be the proper transform of $E$ under the second blow up.
(4) $\tilde{\tilde{X}}_f$ is smooth.
(5) Consider now the action of $\mathbb{C}^*$ on $\mathbb{P}^3$ given by
$$t \cdot (z_0, z_1, z_2, z_3) = (t^{\alpha_0} z_0, t^{\alpha_1} z_1, t^{\alpha_2} z_2, t^{\alpha_3} z_3)$$
with $\alpha_j \in \mathbb{Z}$ and $\alpha_0 + \cdots + \alpha_3 = 0$, thus $(t \cdot f)(z) = t^{-(\alpha_0 + 2\alpha_1)} z_0 z_1^2 + t^{-(2\alpha_2 + \alpha_3)} z_2^2 z_3 - t^{-(\alpha_2 + 2\alpha_3)} z_2 z_3^2$, and $f$ is semi-invariant if and only if
$$\alpha_0 + 2\alpha_1 = 2\alpha_2 + \alpha_3 = \alpha_2 + 2\alpha_3,$$
hence
$$(\alpha_0, \dots, \alpha_3) = (-7\beta, 5\beta, \beta, \beta),$$
with $\beta \in \mathbb{Z}$, and $f$ has weight $-3\beta$.
(6) The monomials of degree three in the variables $z_0, \dots, z_3$ are a basis of semi-invariants for the fixed $\mathbb{C}^*$-action on $\mathbb{C}[z_0, \dots, z_3]_3$. In particular fix $\beta = -1$, then

the subspace spanned by monomials with weight greater or equal to 4 is

$$V = \mathrm{span}\{z_1^3, z_1^2 z_2, z_1^2 z_3, z_1 z_2^2, z_1 z_2 z_3, z_1 z_3^2\}$$

Thus the hypersurfaces that degenerate to $X_f$ under the fixed $\mathbb{C}^\times$-action (with $\beta = -1$) are of the form

$$X_g = \{f + g = 0\},$$

where $g \in V$.

(7) The general $X_g$ has a rational double point at $p_0 = (1 : 0 : 0 : 0)$ and is smooth elsewhere.

(8) $\tilde{\tilde{X}}_f$ is the flat limit of $\tilde{\tilde{X}}_g$ in $\tilde{\tilde{\mathbb{P}}}^3$.

Thus $\tilde{\tilde{X}}_f$ is the central fiber of the degeneration of $\tilde{\tilde{X}}_g$ given by the action of $\mathbb{C}^*$. Denoted by $\pi : \tilde{\tilde{\mathbb{P}}}^3 \to \mathbb{P}^3$ the blow-up, let $A_r$ be the restriction of $\pi^* \mathcal{O}_{\mathbb{P}^3}(r) \otimes \mathcal{O}\left(-\sum_{\ell=0}^4 E_j\right)$ to $\tilde{\tilde{X}}_f$. By Theorem 2.6, $F(\tilde{\tilde{X}}_f, A_r)$ has the same sign of $F(X_f, \mathcal{O}_{X_f}(1))$ when $r$ is large enough. But thanks to [6] (see also [16]), with our sign convention, $(X_f, \mathcal{O}_{X_f}(1))$ has positive Futaki invariant, then the degeneration is $K$-unstable.

## 4. CM line bundle

We start recalling the definition of the (refined) CM-line bundle of a family given by Paul-Tian [18].

Let $f : X \to B$ be a familiy of $n$-dimensional projective schemes. By the term "family" we will always mean that $f$ is a flat projective morphism and more precisely we are given an embedding $i : X \hookrightarrow \mathbb{P}^N \times B$ such that $f = pr_B \circ i$. Let $L = i^* \circ pr_{\mathbb{P}^N}^* \mathcal{O}_{\mathbb{P}^N}(1)$ be the restriction to $X$ of the obvious relatively (very) ample line bundle on $\mathbb{P}^N \times B$. Thanks to the relative ampleness of $L$, the comology of the fiber $H^0(X_b, L_b^k)$ is isomorphic to the fiber over $b \in B$ of the direc image $f_*(L^k)$, at least as $k \gg 0$. With this assumption, by flatness hypothesis the direct image $f_*(L^k)$ is a locally free sheaf on $B$ [7, Proposition 7.9.13], moreover by relative ampleness we have $H^q(X_b, L_b^k) = 0$ for all $q > 0$ so that the fiber $f_*(L^k)_b$ is naturally isomorphic to $H^0(X_b, L_b^k)$ [12, Theorem 12.11]. Thus we conclude that

$$\chi(X_b, L_b^k) = \mathrm{rank}\, f_*(L^k) \tag{3}$$

$$\mu(X_b, L_b^k) = \det f_*(L^k)|_b, \tag{4}$$

for all $b \in B$ and $k \gg 0$. Since $\chi(X_b, L_b^k)$ is a polynomial we have an expansion

$$\mathrm{rank}\, f_*(L^k) = a_0 k^n + a_1 k^{n-1} + \cdots + a_n, \quad \text{as } k \gg 0. \tag{5}$$

Now consider the determinant of the locally free sheaf $f_*(L^k)$ for $k$ big enough. As an easy corollary of a result due to Knudsen and Mumford [14, Proposition 4], we have

$$\det f_*(L^k) = \mu_0^{k^{n+1}} \otimes \mu_1^{k^n} \otimes \cdots \otimes \mu_{n+1}, \tag{6}$$

where $\mu_0, \ldots, \mu_{n+1}$ are $\mathbb{Q}$-line bundles on $B$. Combining (6) and (5), always for $k \gg 0$, we get the asymptotic expansion

$$\det f_*(L^k)^{\frac{1}{\operatorname{rank}(X,L^k)}} = \mu_0^{\frac{k}{a_0}} \otimes \left(\mu_1^{a_0} \otimes \mu_0^{-a_1}\right)^{\frac{1}{a_0^2}} \otimes O(\frac{1}{k}).$$

Up to the factor $-2a_0(n+1)!$ the CM-line associated to the family $(X, L) \to B$ defined by Paul and Tian [18] is the $\mathbb{Q}$-line bundle on $B$ given by the degree zero term of the expansion above

$$\lambda_{\mathrm{CM}}(X, L) = \left(\mu_1^{a_0} \otimes \mu_0^{-a_1}\right)^{\frac{1}{a_0^2}}.$$

Next we want to consider line bundles on $X$ which are not necessarily relatively ample. Although this adds no conceptual difficulties, we need heavy machinary to treat this case which will be fully explained in the appendix.
As above, let $f : X \to B$ be a family of $n$-dimensional projective schemes with $n \geq 1$ and let $L$ be a line bundle on $X$. Since $f$ is projective and $L$ can be condidered as perfect complex of sheaves on $X$ supported in degree zero, then the Euler charcteristic of $L^k$ restricted to a fiber of $f$ is independent of the choosen fiber and is equal to the rank $\operatorname{rank} Rf_*(L^k)$ of the derived push-forward of $L^k$, thus we have the polynomial expansion

$$\operatorname{rank} Rf_*(L^k) = a_0 k^n + a_1 k^{n-1} + \cdots + a_n, \tag{7}$$

with $a_i \in \mathbb{Q}$. In the following we will be interested mainly in line bundles for which the $a_0$-term in the polynomial expansion above is non-zero ($a_0 \neq 0$). As is well known, in particular this hypothesis is verified when $L$ is relatively ample or merely relatively big and nef.
Analogously, the determinant $\det Rf_*(L^k)$ of the derived push-forward of $L^k$ has a *polynomial* expansion in terms of some fixed line bundles on the base $B$. More precisely the following holds [14, Proposition 4]

**Theorem 4.1** (Knudsen-Mumford)**.** *There are line bundles $\nu_i$ on $B$, depending on $f$ and $L$, such that*

$$\det Rf_*(L^k) = \nu_0^{\binom{k}{n+1}} \otimes \nu_1^{\binom{k}{n}} \otimes \cdots \otimes \nu_{n+1}.$$

This clearly implies the existence of $\mathbb{Q}$-line bundles $\mu_i$ on $B$ such that

$$\det Rf_*(L^k) = \mu_0^{k^{n+1}} \otimes \mu_1^{k^n} \otimes \cdots \otimes \mu_{n+1}. \tag{8}$$

To define the CM-line bundle of the given family, consider the following expansion coming from (8) and (7) as $k \to +\infty$

$$\det Rf_*(L^k)^{\frac{1}{\operatorname{rank} Rf_*(L^k)}} = \mu_0^{\frac{k}{a_0}} \otimes \left(\mu_1^{a_0} \otimes \mu_0^{-a_1}\right)^{\frac{1}{a_0^2}} \otimes O(\frac{1}{k}). \tag{9}$$

**Definition 4.2.** In the situation above, the CM-line associated to the family $(X, L)$ is the $\mathbb{Q}$-line bundle on $B$ given by

$$\lambda_{\mathrm{CM}}(X, L) = \left(\mu_1^{a_0} \otimes \mu_0^{-a_1}\right)^{\frac{1}{a_0^2}}.$$

*Remark* 4.3. Cleary, $\lambda_{\mathrm{CM}}(X, L)$ depends on the morphism $f$ and the base $B$ as well. If it is not clear from the context, we shall denote the CM-line bundle by $\lambda_{\mathrm{CM}}(X/B, L)$.

We collect in the next proposition the main properties of the CM-line bundle.

**Proposition 4.4.** *In the situation above we have*

1. $\lambda_{\mathrm{CM}}(X, L^r) = \lambda_{\mathrm{CM}}(X, L)$ *for all* $r > 0$,
2. *if* $\Lambda$ *is a line bundle on* $B$, *then* $\lambda_{\mathrm{CM}}(X, L \otimes f^*\Lambda) = \lambda_{\mathrm{CM}}(X, L)$,
3. *if* $f' : X' \to B$ *is another flat family endowed with a relatively ample line bundle* $L'$, *then*
$$\lambda_{\mathrm{CM}}(X \times_B X', L \boxtimes L') = \lambda_{\mathrm{CM}}(X, L) \otimes \lambda_{\mathrm{CM}}(X', L'),$$
4. *if* $\phi : B' \to B$ *is flat and*
$$\begin{tikzcd} X \times_B B' \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f"] \\ B' \arrow[r, "\phi"] & B \end{tikzcd}$$
*is the base change induced by* $\phi$, *then*
$$\lambda_{\mathrm{CM}}(X \times_B B'/B', p^*L) = \phi^*\lambda_{\mathrm{CM}}(X/B, L).$$

*Proof.* Assertion 1 is obvious from (9). Assertions 2 and 3 are proved in [10], but 2 follows readly by (9) and

$$\det Rf_*(L^k \otimes f^*\Lambda^k) = \det Rf_*(L^k) \otimes \Lambda^{k \operatorname{rank} Rf_*(L^k)},$$

where we used projection formula and the identity $\det(F \otimes A) = \det F \otimes A^{\operatorname{rank} F}$, for all vector bundle $F$ and line bundle $A$ on $B$. Finally, to prove 4 we notice that [12] implies $Rg_*(p^*L^k) = \phi^*Rf_*(L^k)$, whence

$$\det Rg_*(p^*L^k)^{\frac{1}{\operatorname{rank} Rg_*(p^*L^k)}} = \phi^* \det Rf_*(L^k)^{\frac{1}{\operatorname{rank} Rf_*(L^k)}},$$

and the thesis follows. $\square$

Moreover, the CM-line bundle has a sort of continuity property w.r.t. the line bundle $L$. More precisely the following holds:

**Proposition 4.5.** *Let $L$ and $N$ be two line bundles on $X$ and suppose $L$ relatively big and nef as above and $N$ relatively ample w.r.t $f : X \to B$. We have*

$$\lambda_{\mathrm{CM}}(X, L^r \otimes N) = \lambda_{\mathrm{CM}}(X, L) \otimes O(\frac{1}{r}) \qquad \text{as } r \to \infty.$$

*Proof.* By assertion 2 of proposition 4.4 we have

$$\lambda_{\mathrm{CM}}(X, L^r \otimes N) = \lambda_{\mathrm{CM}}(X, L^r \otimes N \otimes f^*\Lambda)$$

for all line bundle $\Lambda$ on $B$, thus we may assume without loss of generality $N$ to be ample on $X$.

For each $k \gg 0$, let $\sigma_1, \ldots, \sigma_k \in H^0(X, N^k)$ be general sections transversal to $f$. Denoted by $Z_i$ the null scheme of $\sigma_i$, we have the exact sequence

$$0 \to L^{rk} \to (L^r \otimes N)^k \to \bigoplus_{i=1}^{k} (L^r \otimes N)^k \otimes \mathcal{O}_{Z_i} \to \bigoplus_{1 \le i_0 < i_1 \le k} (L^r \otimes N)^k \otimes \mathcal{O}_{Z_{i_0} \cap Z_{i_1}} \to \cdots$$
$$\cdots \to \bigoplus_{1 \le i_0 < \cdots < i_n \le k} (L^r \otimes N)^k \otimes \mathcal{O}_{Z_{i_0} \cap \cdots \cap Z_{i_n}} \to 0,$$

whence we get

$$\operatorname{rank} Rf_* \left( L^{rk} \otimes N^k \right) = \operatorname{rank} Rf_*(L^{rk}) +$$
$$+ \sum_{\ell=0}^{n} (-1)^\ell \operatorname{rank} Rf_* \left( \bigoplus_{1 \le i_0 < \cdots < i_\ell \le k} (L^r \otimes N)^k \otimes \mathcal{O}_{Z_{i_0} \cap \cdots \cap Z_{i_\ell}} \right)$$

and

$$\det Rf_* \left( L^{rk} \otimes N^k \right) = \det Rf_*(L^{rk}) \otimes$$
$$\otimes \bigotimes_{\ell=0}^{n} \left( \det Rf_* \left( \bigoplus_{1 \le i_0 < \cdots < i_\ell \le k} (L^r \otimes N)^k \otimes \mathcal{O}_{Z_{i_0} \cap \cdots \cap Z_{i_\ell}} \right) \right)^{(-1)^\ell}.$$

Since

$$\operatorname{rank} Rf_* \left( \bigoplus_{i_0=0}^{k} (L^r \otimes N)^k \otimes \mathcal{O}_{Z_{i_0}} \right) = c_0 r^{n-1} k^n + O(r^{n-2}),$$

$$\operatorname{rank} Rf_*\left(\bigoplus_{1\le i_0\le\cdots\le i_\ell\le k}(L^r\otimes N)^k\otimes\mathcal{O}_{Z_{i_0}\cap\cdots\cap Z_{i_\ell}}\right)=O(r^{n-2})\qquad\text{for all } \ell\ge 1,$$

and analogously

$$\det Rf_*\left(\bigoplus_{i_0=0}^{k}(L^r\otimes N)^k\otimes\mathcal{O}_{Z_{i_0}}\right)=\rho_0 r^n k^{n+1}+O(r^{n-1}),$$

$$\det Rf_*\left(\bigoplus_{1\le i_0\le\cdots\le i_\ell\le k}(L^r\otimes N)^k\otimes\mathcal{O}_{Z_{i_0}\cap\cdots\cap Z_{i_\ell}}\right)=O(r^{n-1})\qquad\text{for all } \ell\ge 1,$$

we have

$$\operatorname{rank} Rf_*(L^{rk}\otimes N^k)=(a_0r^n+c_0r^{n-1})k^n+a_1r^{n-1}k^{n-1}+O(r^{n-2}),$$

$$\det Rf_*(L^{rk}\otimes N^k)=\left(\nu_0^{r^{n+1}}\otimes\rho_0^{r^n}\right)^{k^{n+1}}\otimes\nu_1^{r^nk^n}\otimes O(r^{n-1}).$$

Thus

$$\begin{aligned}\lambda_{\mathrm{CM}}(X,L^r\otimes N) &= \left(\nu_1^{a_0r^{2n}+b_0r^{2n-1}}\otimes\nu_0^{-a_1r^{2n}}\otimes\rho_0^{-a_1r^{2n-1}}\right)^{\frac{1}{(a_0r^n-b_0r^{n-1})^2}}\\ &= \left(\nu_1^{a_0}\otimes\nu_0^{-a_1}\right)^{\frac{1}{a_0^2}}\otimes O\left(\frac{1}{r}\right),\end{aligned}$$

and we are done. □

## 5. Applications

In this section we suppose that the polarized family $f:(X,L)\to B$ of the previous section is a test configuration for a smooth manifold as defined by Donaldson [8]. This means that $B=\mathbb{C}$ and we are given a $\mathbb{C}^*$-action on $X$ that linearizes to $L$ and covers the standard action on $\mathbb{C}$, making $f$ an equivariant map. Moreover the fiber $X_t=f^{-1}(t)$ is smooth for all $t\neq 0$. In this situation the expansion (8) holds in the sense of linearized $\mathbb{Q}$-line bundles, thus the CM-line bundle $\lambda_{\mathrm{CM}}(X,L)$ comes equipped with a linearization. Moreover, proposition 4.4 holds, *mutatis mutandis*, in the sense of linearized line bundles; in particular, property 2 implies that the linearization on $\lambda_{\mathrm{CM}}(X,L)$ is independet of the one chosen on $L$. The central fiber $(X_0,L_0)=(f^{-1}(0),L|_{f^{-1}(0)})$ is equipped with a $\mathbb{C}^*$-action, since it lies over the fixed point $0\in\mathbb{C}$. In case of $L$ ample, the relation between the CM-line bundle and the Donaldson-Futaki invariant $F(X_0,L_0)$ is given by the following [18]

**Proposition 5.1** (Paul-Tian)**.** *The weight of the $\mathbb{C}^*$-action induced on the fiber of $\lambda_{\mathrm{CM}}(X,L)$ over $0\in\mathbb{C}^*$ equals the Donaldson-Futaki invariant $F(X_0,L_0)$ of the central fiber.*

The same remains true if $L$ is not necessarily relatively ample

**Proposition 5.2.** *Let $L$ be a relatively big and nef line bundle on $X$. The weight of the $\mathbb{C}^*$-action induced on the fiber of $\lambda_{\mathrm{CM}}(X, L)$ over $0 \in \mathbb{C}^*$ equals the Donaldson-Futaki invariant $F(X_0, L_0)$ (as defined in 2.1) of the central fiber.*

*Proof.* Since $L$ is $\mathbb{C}^*$-linearized, the determinant $\det Rf_*(L^k)$ inherits a $\mathbb{C}^*$-linearization. Regarding $L$ as a perfect complex (supported on degree zero) on $X$, by Theorem 6.35 we get an equivariant isomorphism

$$\mu(X_0, L_0^k) \simeq \det Rf_*(L^k)|_0$$

for each $k > 0$.
By (8) we get an equivariant expansion

$$\mu(X_0, L_0^k) \simeq \mu_0|_0^{k^{n+1}} \otimes \mu_1|_0^{k^n} \otimes \cdots \otimes \mu_{n+1}|_0,$$

whose weight must coincide for every $k$ with

$$w(X_0, L_0^k) = b_0 k^{n+1} + b_1 k^n + \cdots + b_{n+1},$$

given by Corollary 6.23. Hence the weight on the $\mathbb{Q}$-line $\mu_j|_0$ is $b_j$ and the thesis follows by definition 4.2. □

Combinig the Proposition above with 4.5 we get the following

**Corollary 5.3.** *Let $L, A$ be linearized line bunldes on a scheme $V$ acted on by $\mathbb{C}^*$. Suppose that $L$ is big and nef and $A$ ample. We have*

$$F(V, L^r \otimes A) = F(V, L) + O\left(\frac{1}{r}\right), \qquad \text{as } r \to \infty.$$

Before stating our main result we need to recall two important results. The first one is essentially due to Mumford [13]

**Theorem 5.4** (Equivariant semi-stable reduction)**.** *Let $f : X \to \mathbb{C}$ be a $\mathbb{C}^*$-equivariant family of projective schemes with smooth general fiber. Then there exist an integer $d > 0$ and a projective equivariant morphism $\beta$ as follows*

$$\begin{tikzcd}
X' \arrow[dr, "\beta"] \arrow[ddr, dotted, "f'"'] & & \\
& X \times_{\pi_d} \mathbb{C} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \mathbb{C} \arrow[r, "\pi_d"] & \mathbb{C}
\end{tikzcd}$$

*where $\pi_d(z) = z^d$, such that*

- *$\beta$ is the blow-up of an invariant ideal sheaf supported over $0 \in \mathbb{C}$,*
- *the square is equivariant if we compose the given $\mathbb{C}^*$-action on $f : X \to \mathbb{C}$ with the $d$-fold covering $t \mapsto t^d$ on $\mathbb{C}^*$.*
- *$X'$ is smooth and the central fibre $f'^{-1}(0)$ is a reduced with non-singular components crossing normally.*

*Proof.* For the time being, let us neglect the $\mathbb{C}^*$-action. Applying Mumford's semi-stable reduction theorem [13], we get a smooth curve $C'$ with a marked point $0'$, a finite morphism $\pi : C' \to \mathbb{C}$ such that $\pi^{-1}(0) = \{0'\}$, and a projective morphism $\beta$ as follows

$$\begin{tikzcd} X' \arrow[dr, "\beta"] \arrow[ddr, dotted, "f'"'] & & \\ & X \times_\pi \mathbb{C} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & C' \arrow[r, "\pi"] & \mathbb{C} \end{tikzcd}$$

such that $\beta$ is an isomorphism over $C' \setminus \{0'\}$, $X'$ is smooth and the fiber $f'^{-1}(0')$ is reduced with non-singular components crossing normally.
Now we show that everything can be supposed $\mathbb{C}^*$-equivariant. First of all, restricting $\pi$ to a local chart $\mathbb{C}$ centered at $0' \in C'$, we can suppose without loss of generality that $\pi = \pi_d$ for some integer $d > 0$. Then if we compose the given action on $X$ and $\mathbb{C}$ with the $d$-th covering $t \mapsto t^d$ of $\mathbb{C}^*$, we get a new action on $f : X \to \mathbb{C}$ inducing an action on the fiber product $X \times_{\pi_d} \mathbb{C}$ that makes the projections over $X$ and $\mathbb{C}$ equivariant. Finally, since the existence of $\beta$ is a consequence of the Hironaka's resolution theorem, we can suppose $X'$ acted on by $\mathbb{C}^*$ and $\beta$ equivariant thanks to the equivariant resolution theorem (cf. [15], 4.1 pg.4). $\square$

The second result we need is the following proposition closely related to [19, Proposition 5.1] by Ross-Thomas.

**Proposition 5.5.** *Given a test configuration $f : (X, L) \to \mathbb{C}$ as above, let $f' : (X', L') \to \mathbb{C}$ be another flat equivariant familty and let $\beta : (X', L') \to (X, L)$ be a $\mathbb{C}^*$-equivariant birational map such that $f' = f \circ \beta$ and $L' = \beta^* L$. Then we have*

$$F(X'_0, L'_0) \geq F(X_0, L_0),$$

*with strict inequality if and only if the support of $\beta_* \mathcal{O}_{X'}/\mathcal{O}_X$ has codimension one in $X$.*

*Proof.* Since our definition of Futaki invariant involves higer cohomology, the statement is not the same as the one by Ross-Thomas (loc. cit.). On the other hand we prove the statement reducing to the situation considered by Ross-Thomas. Let

$$X' \xrightarrow{q} Z \xrightarrow{p} X$$

be the Stein factorization of $\beta$, so that $\beta = p \circ q$, $p$ is a finite map, and $q$ is projective, surjective, has connected fibres and $\mathcal{O}_Z \simeq q_*\mathcal{O}_{X'}$. By construction, $Z$ is endowed with a $\mathbb{C}^*$-action and $p, q$ are equivariant and clearly $Z$ is flat over $\mathbb{C}$. For each $m \in \mathbb{Z}$ we have

$$\begin{aligned} Rf'_*((L')^m) &= R(f_* \circ \beta_*)(\beta^* L^m) = Rf_* \circ R\beta_*(\beta^* L^m) \\ &= Rf_*(R\beta_*(\mathcal{O}_{X'}) \otimes L^m) = Rf_*(Rp_* \circ Rq_*(\mathcal{O}_{X'}) \otimes L^m) \\ &= Rf_*(Rp_*(\mathcal{O}_Z) \otimes L^m), \end{aligned}$$

where we used the projection formula and the equivariant isomorphism $\mathcal{O}_Z \simeq q_*\mathcal{O}_{X'}$. Now, by [14, Proposition 8]

$$\det Rf'_*((L')^m) = \bigotimes_{h,k} \det(R^h f_*(H^k(Rp_*(\mathcal{O}_Z) \otimes L^m)))^{(-1)^{h+k}}$$

and thanks to ampleness of $L$ and finiteness of $p$, for $m \gg 0$ finally we get

$$\det Rf'_*((L')^m) = \det f_*(p_*\mathcal{O}_Z \otimes L^m),$$

and we are in the situation considered by Ross-Thomas. In particular, posed $n + 1 = \dim X$, the weight of the $\mathbb{C}^*$-action on

$$\left(\det Rf'_*((L')^m) \otimes \det f_*(L^m)^{-1}\right)|_0$$

is equal to $a\, m^n + O(m^{n-1})$ with $a > 0$ when $supp(\beta_*\mathcal{O}_{X'}/\mathcal{O}_X)$ has dimension $n$ and $a = 0$ otherwise. □

In particular, we can control the behaviour of the Futaki invariant under some important class of birational morphisms.

**Corollary 5.6.** *In the situation of proposition 5.5 we have*

(1) *If $f : X' \to X$ is the blow-up of $X$ along an invariant subscheme of codimension at least two supported (set-theoretically) over the central fiber $X_0$, we have $F(X'_0, L'_0) = F(X_0, L_0)$.*
(2) *If $X_{\mathrm{non-normal}}$ (the set of points where $X$ is not normal) has codimension at least two and $X'$ is normal, then $F(X'_0, L'_0) = F(X_0, L_0)$.*
(3) *If $X_{\mathrm{non-normal}}$ has codimension one and $X'$ is normal, then $F(X'_0, L'_0) > F(X_0, L_0)$.*

*Proof.* The first assertion follows easily after noting that $\beta_*\mathcal{O}_{X'}/\mathcal{O}_X$ can be non-zero only over the center of the blow-up, that has at least codimension two.
To prove assertion two and three observe that thanks to the normality of $X'$, $\beta$ factorizes $\beta = \nu\circ\tilde{\beta}$ through the normalization $\nu : N \to X$. By Zariski's main theorem $\tilde{\beta}_*\mathcal{O}_{X'} \simeq \mathcal{O}_N$, thus $F(X'_0, L'_0) = F(N_0, \nu^* L|_0)$ and $\beta_*\mathcal{O}_{X'} \simeq \nu_*\mathcal{O}_N$, whence the thesis follow. □

*Remark* 5.7. We recall that with our sign convention a test configuration destabilises the general fiber if the Donaldson-Futaki invariant is *positive*. Thus the Proposition above says that we can construct less stable test configurations dominating a given one and

taking a polarization sufficiently close to the pull-back of the original one. This is made more precise in the following

**Theorem 5.8.** *Let $(X, L) \to \mathbb{C}$ be a test configuration for a smooth polarized manifold with $F(X_0, L_0) > 0$, then there is a test configuration $(X', L') \to \mathbb{C}$ for the same polarized manifold with smooth $X'$, whose central fibre is a reduced simple normal crossing divisor, and $F(X'_0, L'_0) > 0$.*

*Proof.* To begin with consider the CM-line bundle $\lambda_{\mathrm{CM}}(X, L)$ and apply the semi-stable reduction theorem 5.4 to the family $(X, L)$. Since $pr_X : X \times_{\pi_d} \mathbb{C} \to X$ is a finite map, then $pr_X^* L$ is ample; moreover by assertion 4 of proposition 4.4 we get $\lambda_{\mathrm{CM}}(X \times_{\pi_d} \mathbb{C}, pr_X^* L) = \pi_d^* \lambda_{\mathrm{CM}}(X, L)$, thus on the central fibres we have

$$F((X \times_{\pi_d} \mathbb{C})_0, (pr_X^* L)_0) = d\, F(X_0, L_0). \tag{10}$$

On the other hand, denoted by $E$ the exceptional divisor of $\beta$, the line bundle $L'(r) = \beta^* pr_X^* L^r(-E)$ on $X'$ is relatively ample for $r$ big enough and $E$ is trivial outside form central fibre, thus $(X', L'(r)) \to \mathbb{C}$ is a test configuration for the original polarized manifold. By proposition 5.2 we can approximate the Donaldson-Futaki invariant of the line bundle pulled-back via $\beta$

$$F(X'_0, L'(r)_0) = F(X'_0, (\beta^* pr_X^* L)_0) + O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty, \tag{11}$$

but finally we observe that

$$\beta : (X', \beta^* pr_X^* L) \to (X \times_{\pi_d} \mathbb{C}, pr_X^* L)$$

satisfies the hypothesis of proposition 5.5 thus combining with (10) and (11), for each $\varepsilon > 0$ we get $r_0(\varepsilon)$ such that

$$F(X'_0, L'(r)_0) > d\, F(X_0, L_0) - \varepsilon \quad \text{for all } r > r_0(\varepsilon).$$

Since $X'$ is smooth and $X'_0$ is a reduced simple normal crossing divisor by semi-stable reduction theorem and $F(X_0, L_0)$ is positive by hypothesis the thesis follows choosing $\varepsilon$ sufficiently small. □

Adding an hypothesis on the dimension of the non-normal locus of $X$, thanks to third assertion of corollary 5.6 we can remove the hypothesis on the sign of the Futaki invariant

**Theorem 5.9.** *Let $(X, L) \to \mathbb{C}$ be a test configuration for a smooth polarized manifold and assume that $X_{\mathrm{non-normal}}$ has codimension one. Then there is a test configuration $(X', L') \to \mathbb{C}$ for the same polarized manifold with smooth $X'$, whose central fibre is a reduced simple normal crossing divisor, and*

$$F(X'_0, L'_0) > d\, F(X_0, L_0)$$

*for some integer $d > 0$.*

*Proof.* The situation is almost identical to the one of theorem 5.8, but now we can avoid the final $(\varepsilon, \delta)$-argument. Indeed, $\beta$ factorizes through the normalization and by third assertion of corollary 5.6 we get the inequality

$$F(X_0', \beta^* pr_X^* L_0) > d\, F(X_0, L_0),$$

and the thesis follows. □

As mentioned in the introduction, it si necessary to define some normalization on the space of test configurations which kills the trivial ways to increase arbitrarily the Futaki invariant in a geometric meaningless way. Futaki-Mabuchi [11] in the smooth case, Székelyhidi [22] and Donaldson [9] for general schemes have solved this problem by defining a norm (which is denoted by $N_2$ in [9]) of a test configuration $(X, L) \to \mathbb{C}$ as

$$N_2(X, L) = \sqrt{Q - \frac{b_0^2}{a_0}}\ ,$$

where $Q$ is the leading coefficient of the expansion in $k$ of $\operatorname{tr} A_k^2$ (see definition 2.1) referred of course to the action on the central fiber $(X_0, L_0)$.

**Corollary 5.10.** *In the situation of theorem 5.9, we have*

$$\Psi(X', L') > \Psi(X, L).$$

*Proof.* By definition

$$\Psi(X, L) = \frac{F(X, L)}{N_2(X, L)}.$$

Performing a base change $t \to t^d$ transforms $A_k$ to $d\, A_k$, whence $N_2(X, L)$ changes to $d\, N_2(X, L)$. On the other hand, since $N_2(X, L)$ depends only on leading asymptotic coefficients, thanks to the proof of Proposition 5.5, it is unchanged under the birational map $(X', \beta^* pr_X^* L^r) \to (X, L)$ given by equivariant semi-stable reduction. Finally the continuity property for $a_0$, $b_0$ and $Q$ implies

$$N_2(X', \beta^* pr_X^* L^r(-E)) = N_2(X', \beta^* pr_X^* L^r) + O\left(\frac{1}{r}\right)$$

and the thesis follows easily by continuty property of $F(X, L)$. □

In light of the proof above, the one of Theorem 1.4 is reduced to a straightforward exercise.

## 6. Appendix. Knudsen-Mumford functors

### 6.1. Picard categories.

**Definition 6.1.** A *monoidal symmetric category* $C = C(\otimes, e, \alpha, \gamma, \lambda, \rho)$ is a category endowed with a bifunctor $\otimes : C \times C \to C$, a fixed object $e$ and four natural isomorphisms

$$\begin{aligned} \alpha(a,b,c) &: (a \otimes b) \otimes c \to a \otimes (b \otimes c), \\ \gamma(a,b) &: a \otimes b \to b \otimes a, \\ \lambda(a) &: e \otimes a \to a, \\ \rho(a) &: a \otimes e \to a, \end{aligned}$$

satisfying suitable coherence axioms [17].

Thanks to the coherence axioms we can, up to natural isomorphisms, drop parenthesis and copies of $e$ in products, and treat $\otimes$ as if it were commutative.

**Definition 6.2.** A *right inverse* of an object $a$ in a monoidal symmetric category $C = C(\otimes, e, \alpha, \gamma, \lambda, \rho)$ is an object $b$ and an isomorphism $\iota_r : a \otimes b \to e$. To any right inverse there is associated a left inverse $\iota_\ell = \iota_r \circ \gamma(b, a) : b \otimes a \to e$. An object admitting an inverse is called *invertible*.

**Definition 6.3.** A *Picard category* is a monoidal symmetric category with the properties that every object is invertible and every morphism is an isomorphism. In each Picard category we suppopse fixed an *inverse structure*, i.e. the choice, for any given object $a$, of an inverse $a^{-1}$ with an isomorphism $a \otimes a^{-1} \simeq e$. In the following we suppose that each functor $F : C_1 \to C_2$ between Picard categories $C_1 = C_1(\otimes_1, e_1, \alpha_1, \gamma_1, \lambda_1, \rho_1)$ and $C_2 = C_2(\otimes_2, e_2, \alpha_2, \gamma_2, \lambda_2, \rho_2)$ preserves the structure of Picard Category. More precisely $F$ comes with isomorphisms

$$\begin{aligned} F \circ \otimes_1 &\xrightarrow{\sim} \otimes_2 \circ F, \\ F \circ \alpha_1 &\xrightarrow{\sim} \alpha_2 \circ F \\ F \circ \gamma_1 &\xrightarrow{\sim} \gamma_2 \circ F \\ F(e_1) &\xrightarrow{\sim} e_2 \\ F \circ \lambda_1 &\xrightarrow{\sim} \lambda_2 \circ F \\ F \circ \rho_1 &\xrightarrow{\sim} \rho_2 \circ F \end{aligned}$$

satisfying obvious compatibility conditions.

*Remark* 6.4. Loosely speaking a Picard category works like an abelian group (with multiplicative notation), but in general the class of its objects is not a set. After setting $a^0 = e$, with notation introduced is well defined up to natural isomorphism the object $a^m$ for all $m \in \mathbb{Z}$.

*Example* 6.5. The category $Vect(\mathbb{C}, 1)$ of one-dimensional $\mathbb{C}$-vector spaces endowed with the tensor product is a Picard category. Clearly, $\mathbb{C}$ is the unit and the inverse of the vector space $E$ is the dual, which we denote by $E^{-1}$.

*Example* 6.6. Given a reductive complex linear group, the category $Vect(\mathbb{C}, 1, G)$ of one-dimensional $G$-representations is a Picard category. The trivial representation over $\mathbb{C}$ is the unit and the inverse is the contragradient representation.

*Example* 6.7. Given a scheme, or more gererally a ringed space, $(X, \mathcal{O}_X)$, the class $Inv(X)$ of all invertible sheaves on $X$ endowed with the tensor product of $\mathcal{O}_X$-modules and the class of morphisms restricted to isomorphisms is a Picard category. Here $\mathcal{O}_X$ is the unit and the inverse structure is just given by inversion of invertible sheaves.

*Example* 6.8. The group of integers $\mathbb{Z}$ can be tought (not very naturally) as a Picard category $C(\mathbb{Z})$ where the product is the sum of integers and $m$ is isomorphic to $n$ via the isomorphism 'adding $n-m$'. Analogously, each abelian group can be thought of as a Picard category.

6.2. **$\mathbb{Q}$-Picard categories.**

**Definition 6.9.** Given a Picard category $C$, we define the Picard category $C_{\mathbb{Q}}$ as follows

$$\begin{aligned} \text{Ob } C_{\mathbb{Q}} &= \{(a,m) \text{ s.t. } a \text{ is an object of } C \text{ and } m \in \mathbb{Z} \setminus \{0\}\}\,, \\ \text{Hom}\left((a,m),(b,n)\right) &= \bigcup_{\ell>0} \text{Hom}(a^{\ell n}, b^{\ell m}), \\ (a,m)\otimes(b,n) &= (a^n \otimes b^m, mn), \end{aligned}$$

and taking $(e,1)$ as unit and $(a,m)^{-1} = (a^{-1}, m)$ as inverse structure.

*Remark* 6.10. The inverse structure of $C_{\mathbb{Q}}$ is well defined since $(e,1)$ and $(e,m)$ are naturally isomorphic for all $m \in \mathbb{Z} \setminus \{0\}$. More generally $(a^m, m)$ is naturally isomorphic to $(a,1)$ via the identity in $\text{Hom}(a^m, a^m)$.

**Definition 6.11.** An object $a$ of a Picard category $C$ is called a *torsion object* if there is an integer $m_0 \neq 0$ and an isomorphism $a^{m_0} \simeq e$.

*Remark* 6.12. If $a$ is a torsion object as in the definition above, then for each $m \in \mathbb{Z}$ we have the following chain of natural isomorphisms

$$(a^m, m) \simeq (a,1) \simeq (a^{m_0}, m_0) \simeq (e, n_0) \simeq (e,1).$$

*Remark* 6.13. The category $C_{\mathbb{Q}}$ comes with a functor $R : C \to C_{\mathbb{Q}}$ defined by $R(a) = (a,1)$ and $R(f) = f$ for all $f \in \text{Hom}(a,b)$. Clearly $R$ is always faithful and never full (actually if there is at least an object $a$ of $C$ such that $a^2 \neq a$).

*Example* 6.14. The following exaple show why $C_{\mathbb{Q}}$ can be tought as a rational extension of a given Picard category $C$. Given an abelian group $G$ (with additive notation), let $C(G)$ be the Picard category naturally associated to $G$ where

$$\begin{aligned} \mathrm{Ob}\, C(G) &= \{g \in G\}, \\ \mathrm{Hom}(h, g) &= \{g - h\}, \\ h \otimes g &= h + g, \end{aligned}$$

the identity and the inverses are those of the group $G$. Consider the functors

$$\begin{aligned} \mathrm{L} &: C(G)_{\mathbb{Q}} \to C(\mathbb{Q} \otimes_{\mathbb{Z}} G) \\ \Gamma &: C(\mathbb{Q} \otimes_{\mathbb{Z}} G) \to C(G)_{\mathbb{Q}} \end{aligned}$$

defined by $\mathrm{L}(g, m) = \frac{1}{m} \otimes_{\mathbb{Z}} g$ and $\Gamma(\frac{n}{m} \otimes_{\mathbb{Z}} g) = (g^n, m)$. Clearly L and $\Gamma$ realize an equivalence between the categories $C(G)_{\mathbb{Q}}$ and $C(\mathbb{Q} \otimes_{\mathbb{Z}} G)$.

*Example* 6.15. Given a ringed space, or more genrally a scheme, $(X, \mathcal{O}_X)$ we can consider the category $Inv(X)_{\mathbb{Q}}$ of $\mathbb{Q}$-invertible shaves on $X$. We stress that by definition the only morphisms that we have between $\mathbb{Q}$-invertible sheaves are isomorphisms.

*Remark* 6.16. From now on we denote by $a^{\frac{1}{m}}$ the object $(a, m)$ of a $\mathbb{Q}$-Picard category $C_{\mathbb{Q}}$. By remark 6.10 the product in $C_{\mathbb{Q}}$ follows the usual rules of powers.

### 6.3. **Numerical polynomials with values in a Picard category.**

**Definition 6.17.** Given a Picard category $C$, a *$C$-valued numerical function* is a set $\{p(m)\}_{m \in \mathbb{Z}}$ of object of $C$ indexed by $\mathbb{Z}$. A *$C$-valued numerical polynomial* of degree $d$ is a *$C$-valued numerical function* $p(m)$ such that

$$p(m) \simeq p_0^{\binom{m}{d}} \otimes p_1^{\binom{m}{d-1}} \otimes \cdots \otimes p_d,$$

for some fixed objects $p_0, \ldots, p_d$ of $C$.

The following lemma is analogous to [12, Proposition 7.3]

**Lemma 6.18.** *Let $f(m)$ be a $C$-valued numerical function. If there is a $C$-valued numerical polynomial $q(m)$ of degree $d$, and $m_0 \in \mathbb{Z}$ such that*

$$\Delta f(m) = f(m+1) \otimes f(m)^{-1}$$

*is isomorphic to $q(m)$ for all $m \geq m_0$, then there exists a $C$-valued numerical polynomial $p(m)$ of degree $d+1$ such that $f(m) \simeq p(m)$ for all $m \geq m_0$.*

*Proof.* By definition there are objects $q_0, \ldots, q_d$ of $C$ such that

$$q(m) \simeq q_0^{\binom{m}{d}} \otimes q_1^{\binom{m}{d-1}} \otimes \cdots \otimes q_d.$$

Define

$$p(m) = q_0^{\binom{m}{d+1}} \otimes q_1^{\binom{m}{d}} \otimes \cdots \otimes q_d^{\binom{m}{1}}.$$

Then $\Delta p(m) \otimes q(m)^{-1} \simeq e$ for all $m \in \mathbb{Z}$, whence $\Delta(p(m) \otimes f(m)^{-1}) \simeq e$ for all $m \geq m_0$. So there is an object $q_{d+1}$ of $C$ such that

$$f(m) \simeq p(m) \otimes q_{d+1}$$

as required. □

*Remark* 6.19. Each $C$-valued polynomial

$$p(m) = p_0^{\binom{m}{d}} \otimes p_1^{\binom{m}{d-1}} \otimes \cdots \otimes p_d,$$

determines a $C_{\mathbb{Q}}$-valued polynomial, that we denote with the same symbol,

$$p(m) = a_0^{m^d} \otimes a_1^{m^{d-1}} \otimes \cdots \otimes a_d,$$

where the coefficients $a_j$ are a *rational* combination of $p_0, \ldots, p_j$. In particular we have

$$\begin{aligned} a_0 &= p_0^{\frac{1}{d!}}, \\ a_1 &= p_0^{-\frac{d-1}{2(d-1)!}} \otimes p_1^{\frac{1}{(d-1)!}}. \end{aligned}$$

6.4. **The Knudsen-Mumford characteristic.** For a given $n$-dimensional $\mathbb{C}$-vector space $E$, we recall that $\det E = \bigwedge^n E$ with the convention that $\det E = \mathbb{C}$ if $E = 0$.

**Definition 6.20.** Let $X$ be a compact complex scheme and let $\mathcal{F}$ be a coherent sheaf on $X$. We define the *Knudsen-Mumford characteristic* of $F$ by

$$\mu(X, \mathcal{F}) = \bigotimes_{i \geq 0} \det H^i(X, \mathcal{F})^{(-1)^i}.$$

Since every isomorphism $\mathcal{F} \simeq \mathcal{F}'$ induces isomorphisms $H^i(X, \mathcal{F}) \simeq H^i(X, \mathcal{F}')$, we get the following

**Proposition 6.21.** *The Knudsen-Mumford characteristic defines a covariant functor*

$$\mu(X, \cdot) : Coh(X)_{is} \to Vect(\mathbb{C}, 1)$$

*between the category of coherent sheaves on X (and where morphisms between objects are isomorphisms of coherent sheaves) and the Picard category of one-dimensional* $\mathbb{C}$*-vector spaces.*

Moreover, for each short-exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $Coh(X)$ we have the functorial isomorphism

$$\mu(X, \mathcal{F}) \simeq \mu(X, \mathcal{F}') \otimes \mu(X, \mathcal{F}''),$$

and more generally

$$\mu(X,\mathcal{F}_0) \simeq \bigotimes_{j=1}^{k} \mu(X,\mathcal{F}_j)^{(-1)^{j-1}}$$

for each exact sequence $0 \to \mathcal{F}_0 \to \cdots \to \mathcal{F}_k \to 0$. These results are rather direct consequences of the long exact cohomology sequence and elementary properties of tensor products.
Analogously to the Euler characteristic, on a projective scheme $X \subset \mathbb{P}^N$ the Knudsen-Mumford characteristic of a twisted sheaf is a $Vect(\mathbb{C},1)$-valued polynomial in the degree of the twisting.

**Theorem 6.22.** *For each coherent sheaf $\mathcal{F}$ on $X \subset \mathbb{P}^N$ with $n$-dimensional support, there exist one-dimensional $\mathbb{C}$-vector spaces $\mu_j = \mu_j(\mathcal{F})$ such that*

$$\mu(X,\mathcal{F}(m)) = \mu_0^{\binom{m}{n+1}} \otimes \mu_1^{\binom{m}{n}} \otimes \cdots \otimes \mu_{n+1}$$

*for all $m \in \mathbb{Z}$.*

*Proof.* This is very similar to the Euler characteristic case [12, Chap. III, Exercise 5.1]. To begin with we notice that if $\mathcal{F} = 0$ is the zero sheaf, then $\mu(X,\mathcal{F}(m)) = \mathbb{C}$. In general, let $S = \operatorname{Supp}\mathcal{F} \subset X$. If $\dim S = 0$, then $H^i(X,\mathcal{F}(m)) = 0$ for all $i > 0$ by Grothendieck's vanishing, thus

$$\mu(X,\mathcal{F}(m)) = \det H^0(X,\mathcal{F}(m)) \simeq \bigotimes_{p\in S} \mathcal{F}_p \otimes \mathcal{O}_X(m)_p \simeq \left(\bigotimes_{p\in S} \mathcal{F}_p\right) \otimes \left(\bigotimes_{p\in S} \mathcal{O}_X(1)_p\right)^m,$$

and the theorem is proved in case $n = 0$.
We prove the general case by induction on $n$. If $\sigma : \mathcal{O}_X \to \mathcal{O}_X(1)$ is a section not identically zero on $S$, consider the induced exact sequence

$$0 \to \mathcal{R} \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{Q} \to 0,$$

where $\dim \operatorname{Supp}\mathcal{R}$ and $\dim \operatorname{Supp}\mathcal{Q}$ are less than $n$. Thus, after tensoring by $\mathcal{O}_X(m)$, we get

$$\mu(X,\mathcal{F}(m)) \otimes \mu(X,\mathcal{F}(m-1))^{-1} \simeq \mu(X,\mathcal{Q}(m)) \otimes \mu(X,\mathcal{R}(m))^{-1}.$$

By inductive hypothesis $\mu(X,\mathcal{Q}(m))$, $\mu(X,\mathcal{R}(m))$ have degree at most $n$, thus the theorem follows by lemma 6.18. □

When considering linearized line bundles which are not necessarily ample , by arguing as in Snapper [21] or Cartier [3] one readily gets the following:

**Corollary 6.23.** *Let $V$ be a $n$-dimensional projective scheme endowed with a $\mathbb{C}^*$-action and let $L$ be a nef and big linearized line bundle on $X$. Denoted by $w(V, L^k)$ the weight of the induced action on $\mu(V, L^k)$, we have the polynomial expansion*

$$w(V, L^k) = b_0 k^{n+1} + b_1 k^n + \cdots + b_{n+1}$$

*for suitable $b_j \in \mathbb{Q}$.*

6.5. **Knudsen-Mumford functors.**

**Definition 6.24.** A *Knudsen-Mumford functor* $\varphi = (\Gamma, \varphi, i)$ consists of the assignement to any complex scheme $X$ of the following data:

(1) a Picard category $\Gamma(X)$,
(2) a covariant functor
$$\varphi_X : LF(X)_{is} \to \Gamma(X)$$
defined on the category $LF(X)_{is}$ of finite locally free sheaves on $X$ and isomorphisms;
(3) an isomorphism
$$i_{\varphi_X}(\Sigma) : \varphi_X(\mathcal{E}') \otimes \varphi_X(\mathcal{E}'') \to \varphi_X(\mathcal{E})$$
for each short-exact sequence $\Sigma = \{0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0\}$ in $LF(X)$;

and to any holomorphic map $f : Y \to X$:

(4) a covariant functor of Picard categories
$$\Gamma(f) : \Gamma(X) \to \Gamma(Y),$$
such that $\Gamma(\mathrm{id}) = \mathrm{id}$ and $\Gamma(fg) \simeq \Gamma(g) \circ \Gamma(f)$ for every composite map
$$Z \xrightarrow{g} Y \xrightarrow{f} X\ .$$

We require that:

(1) $\varphi_X(0) = e \in \Gamma(X)$.
(2) For exact sequences of the form
$$\begin{aligned} \Sigma' &= \{0 \to \mathcal{E} \stackrel{\mathrm{id.}}{\to} \mathcal{E} \to 0 \to 0\}, \\ \Sigma'' &= \{0 \to 0 \to \mathcal{E} \stackrel{\mathrm{id.}}{\to} \mathcal{E} \to 0\} \end{aligned}$$
in $LF(X)$, the isomorphisms
$$\begin{aligned} i_{\varphi_X}(\Sigma') &: \varphi_X(\mathcal{E}) \otimes e \to \varphi_X(\mathcal{E}), \\ i_{\varphi_X}(\Sigma'') &: e \otimes \varphi_X(\mathcal{E}) \to \varphi_X(\mathcal{E}) \end{aligned}$$
coincide respectively with $\rho(\varphi_X(\mathcal{E}))$ and $\lambda(\varphi_X(\mathcal{E}))$ in the Picard category $\Gamma(X)$.

(3) $i_{\varphi_X}$ is functorial, i.e. for each isomorphism $\varepsilon : \Sigma \to \mathrm{T}$ of short-exact sequences in $LF(X)$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}' \arrow[r] \arrow[d, "\varepsilon'"] & \mathcal{E} \arrow[r] \arrow[d, "\varepsilon"] & \mathcal{E}'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0
\end{tikzcd}$$

we have the commutative diagram

$$\begin{tikzcd}
\varphi_X(\mathcal{E}') \otimes \varphi_X(\mathcal{E}'') \arrow[r, "i_{\varphi_X}(\Sigma)"] \arrow[d, "\varphi_X(\varepsilon')\otimes\varphi_X(\varepsilon'')"'] & \varphi_X(\mathcal{E}) \arrow[d, "\varphi_X(\varepsilon)"] \\
\varphi_X(\mathcal{F}') \otimes \varphi_X(\mathcal{F}'') \arrow[r, "i_{\varphi_X}(\mathrm{T})"] & \varphi_X(\mathcal{F})
\end{tikzcd}$$

(4) Given morphisms $\eta : \mathrm{P} \to \Sigma$ and $\varepsilon : \Sigma \to \mathrm{T}$ of short-exact sequences that form a square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{D}' \arrow[r, "\eta'"] \arrow[d] & \mathcal{E}' \arrow[r, "\varepsilon'"] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{D} \arrow[r, "\eta"] \arrow[d] & \mathcal{E} \arrow[r, "\varepsilon"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{D}'' \arrow[r, "\eta''"] \arrow[d] & \mathcal{E}'' \arrow[r, "\varepsilon''"] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where also the rows $\mathrm{R}'$, $\mathrm{R}$, $\mathrm{R}''$ are exact, we have the commutative diagram

$$\begin{tikzcd}
\varphi_X(\mathcal{D}') \otimes \varphi_X(\mathcal{D}'') \otimes \varphi_X(\mathcal{F}') \otimes \varphi_X(\mathcal{F}'') \arrow[r, "i_{\varphi_X}(\mathrm{P})\otimes i_{\varphi_X}(\mathrm{T})"] \arrow[d, "i_{\varphi_X}(\mathrm{R}')\otimes i_{\varphi_X}(\mathrm{R}'')"'] & \varphi_X(\mathcal{D}) \otimes \varphi_X(\mathcal{F}) \arrow[d, "i_{\varphi_X}(\mathrm{R})"] \\
\varphi_X(\mathcal{E}') \otimes \varphi_X(\mathcal{E}'') \arrow[r, "i_{\varphi_X}(\Sigma)"] & \varphi_X(\mathcal{E})
\end{tikzcd}$$

(5) For every morphism $f : Y \to X$ we have an isomorphism

$$\zeta(f) : \varphi_Y \circ f^* \to \Gamma(f) \circ \varphi_X,$$

such that

- for any short-exact sequence $\Sigma = \{0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0\}$ in $LF(X)$ we have the commutative diagram

$$\begin{tikzcd}
\varphi_Y(f^*\mathcal{E}') \otimes \varphi_Y(f^*\mathcal{E}'') \arrow[r, "i_{\varphi_Y}(f^*\Sigma)"] \arrow[d, "\zeta(f)\otimes\zeta(f)"'] & \varphi_Y(f^*\mathcal{E}) \arrow[d, "\zeta(f)"] \\
\Gamma(f)\circ\varphi_X(\mathcal{E}') \otimes \Gamma(f)\circ\varphi_X(\mathcal{E}'') \arrow[r, "\Gamma(f)\left(i_{\varphi_X}(\Sigma)\right)"] & \Gamma(f)\circ\varphi_X(\mathcal{E})
\end{tikzcd}$$

- for consecutive morphisms

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

we have a commutative diagram

$$\begin{tikzcd}
\varphi_Z \circ (fg)^* \arrow[r, "\zeta(fg)"] \arrow[d, "\zeta(g)"'] & \Gamma(fg)\circ\varphi_X \\
\Gamma(g)\circ\varphi_Y\circ f^* \arrow[ur, "\Gamma(g)\zeta(f)"'] &
\end{tikzcd}$$

*Example* 6.25. The rank of a finite locally free sheaf over a complex scheme defines a Knudsen-Mumford functor

$$\mathrm{rank} : LF(X)_{is} \to C(\mathbb{Z}),$$

where $\Gamma(X) = C(\mathbb{Z})$ is independent of $X$, and $\Gamma(f) = \mathrm{id}.$ for every morphism $f : X \to Y$.

*Example* 6.26. Given a finite locally free sheaf $\mathcal{E}$ on a complex space, we can consider its *determinant* defined by $\det \mathcal{E} = \bigwedge^{\mathrm{top}} \mathcal{E}$. Thus $\det \mathcal{E}$ is an invertible sheaf on $X$ and it defines a Knudsen-Mumford functor

$$\det : LF(X)_{is} \to Inv(X)_{is},$$

where $\Gamma(X) = Inv(X)_{is}$ is independent of $X$, and $\Gamma(f) = f^*$ for every morphism $f : X \to Y$.

**Definition 6.27.** Let $\mathcal{K}$ be a complex of $\mathcal{O}_X$-modules on a scheme $(X, \mathcal{O}_X)$. $\mathcal{K}$ is called *perfect* if for each $x \in X$ there is an open $U$ and a bounded complex $\mathcal{G}$ of finite free $\mathcal{O}_X|_U$-modules and a quasi-isomorphism

$$\mathcal{G} \to \mathcal{K}|_U.$$

In this situation we say that $\mathcal{G}$ is a local trivialization of $\mathcal{K}$ over $U$ (or around $x$).

*Remark* 6.28. In other words a complex of $\mathcal{O}_X$-modules is perfect if it is locally quasi-isomorphic to a complex of finite free $\mathcal{O}_X$-modules.

**Definition 6.29.** We denote by $P(X)$ the full subcategory of $D(X)$ (the derived category of $Mod(X)$) whose objects are perfect complexes of $\mathcal{O}_X$-modules.

*Remark* 6.30. It is not difficult to show that if $X$ is affine, then each perfect complex on $X$ is globally quasi-isomorphic to a complex of finite free $\mathcal{O}_X$-modules.

*Remark* 6.31. Given a morphism $f : X \to Y$, are defined the direct image functor $f_* : Mod(X) \to Mod(Y)$ and the inverse image functor $f^* : Mod(Y) \to Mod(X)$. As is well known they induce derived functors $Rf_* : D^+(X) \to D^+(Y)$ and $Lf^* : D^-(Y) \to D^-(X)$ between the derived categories of bounded above and bounded below complexes on $X$ and $Y$. If $\mathcal{H}$ is a perfect complex on $Y$, then $Lf^*(\mathcal{H})$ is perfect on $X$. On the other hand, given a perfect complex $\mathcal{K}$ on $X$, the complex $Rf_*(\mathcal{K})$ need not to be perfect.

**Definition 6.32.** A morphism of scheme $f : X \to Y$ is called *perfect* if the derived functor $Rf_*$ restricts to $Rf_* : P(X) \to P(Y)$.

**Theorem 6.33** (Knudsen-Mumford, Knudsen)**.** *Given a Knudsen-Mumford functor* $\varphi = (\Gamma, \varphi, i)$*, for every complex space* $X$ *there is an essentially unique extension (which we denote by the same symbol* $\varphi_X$*)*

$$\begin{tikzcd} P(X)_{is} \arrow[r, dashed] & \Gamma(X) \\ LF(X)_{is} \arrow[u, hook] \arrow[ur, "\varphi_X"'] & \end{tikzcd}$$

*and an isomorphism*

$$i_{\varphi_X}(\Sigma) : \varphi_X(\mathcal{E}') \otimes \varphi_X(\mathcal{E}'') \to \varphi_X(\mathcal{E})$$

*for every distinguished triangle* $\Sigma = \{\mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to \mathcal{E}'[1]\}$ *in* $P(X)$ *satisfying:*

(1) $\varphi_X(0) = e \in \Gamma(X)$.
(2) *For distinguished triangles of the form*

$$\begin{aligned} \Sigma' &= \{\mathcal{E} \xrightarrow{\mathrm{id}} \mathcal{E} \to 0 \to \mathcal{E}[1]\}, \\ \Sigma'' &= \{0 \to \mathcal{E} \xrightarrow{\mathrm{id}} \mathcal{E} \to 0[1]\} \end{aligned}$$

*in* $P(X)$*, the isomorphisms*

$$\begin{aligned} i_{\varphi_X}(\Sigma') &: \varphi_X(\mathcal{E}) \otimes e \to \varphi_X(\mathcal{E}), \\ i_{\varphi_X}(\Sigma'') &: e \otimes \varphi_X(\mathcal{E}) \to \varphi_X(\mathcal{E}) \end{aligned}$$

*coincide respectively with* $\rho(\varphi_X(\mathcal{E}))$ *and* $\lambda(\varphi_X(\mathcal{E}))$ *in* $\Gamma(X)$.

(3) *for each isomorphism* $\varepsilon : \Sigma \to \mathrm{T}$ *of distinguished triangles in* $P(X)$

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "\varepsilon'"] & \mathcal{E} \arrow[r] \arrow[d, "\varepsilon"] & \mathcal{E}'' \arrow[r] \arrow[d, "\varepsilon''"] & \mathcal{E}'[1] \arrow[d, "\varepsilon'[1]"] \\
\mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & \mathcal{F}'[1]
\end{tikzcd}$$

*we have the commutative diagram*

$$\begin{tikzcd}
\varphi_X(\mathcal{E}') \otimes \varphi_X(\mathcal{E}'') \arrow[r, "i_{\varphi_X}(\Sigma)"] \arrow[d, "\varphi_X(\varepsilon')\otimes\varphi_X(\varepsilon'')"'] & \varphi_X(\mathcal{E}) \arrow[d, "\varphi_X(\varepsilon)"] \\
\varphi_X(\mathcal{F}') \otimes \varphi_X(\mathcal{F}'') \arrow[r, "i_{\varphi_X}(\mathrm{T})"] & \varphi_X(\mathcal{F})
\end{tikzcd}$$

(4) *Given morphisms* $\eta : \mathrm{P} \to \Sigma$ *and* $\varepsilon : \Sigma \to \mathrm{T}$ *of distinguished triangles in* $P(X)$ *that form a square*

$$\begin{tikzcd}
\mathcal{D}' \arrow[r, "\eta'"] \arrow[d] & \mathcal{E}' \arrow[r, "\varepsilon'"] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{D}'[1] \arrow[d] \\
\mathcal{D} \arrow[r, "\eta"] \arrow[d] & \mathcal{E} \arrow[r, "\varepsilon"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{D}[1] \arrow[d] \\
\mathcal{D}'' \arrow[r, "\eta''"] \arrow[d] & \mathcal{E}'' \arrow[r, "\varepsilon''"] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & \mathcal{D}''[1] \\
\mathcal{D}'[1] \arrow[r, "\eta'[1]"] & \mathcal{E}'[1] \arrow[r, "\varepsilon'[1]"] & \mathcal{F}'[1] &
\end{tikzcd}$$

*where also the rows* $\mathrm{R}'$, $\mathrm{R}$, $\mathrm{R}''$ *are distinguished triangles, we have the commutative diagram*

$$\begin{tikzcd}
\varphi_X(\mathcal{D}') \otimes \varphi_X(\mathcal{D}'') \otimes \varphi_X(\mathcal{F}') \otimes \varphi_X(\mathcal{F}'') \arrow[r, "i_{\varphi_X}(\mathrm{P})\otimes i_{\varphi_X}(\mathrm{T})"] \arrow[d, "i_{\varphi_X}(\mathrm{R}')\otimes i_{\varphi_X}(\mathrm{R}'')"'] & \varphi_X(\mathcal{D}) \otimes \varphi_X(\mathcal{F}) \arrow[d, "i_{\varphi_X}(\mathrm{R})"] \\
\varphi_X(\mathcal{E}') \otimes \varphi_X(\mathcal{E}'') \arrow[r, "i_{\varphi_X}(\Sigma)"] & \varphi_X(\mathcal{E})
\end{tikzcd}$$

(5) *For every holomorphic map* $f : Y \to X$ *we have an isomorphism*

$$\zeta(f) : \varphi_Y \circ Lf^* \to \Gamma(f) \circ \varphi_X,$$

*such that*

- *for any distinguished triangle* $\Sigma = \{\mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to \mathcal{E}'[1]\}$ *in* $P(X)$ *we have the commutative diagram*

$$\begin{tikzcd}
\varphi_Y(Lf^*\mathcal{E}') \otimes \varphi_Y(Lf^*\mathcal{E}'') \arrow[r, "i_{\varphi_Y}(Lf^*\Sigma)"] \arrow[d, "\zeta(f)\otimes\zeta(f)"'] & \varphi_Y(Lf^*\mathcal{E}) \arrow[d, "\zeta(f)"] \\
\Gamma(f) \circ \varphi_X(\mathcal{E}') \otimes \Gamma(f) \circ \varphi_X(\mathcal{E}'') \arrow[r, "\Gamma(f)(i_{\varphi_X}(\Sigma))"] & \Gamma(f) \circ \varphi_X(\mathcal{E})
\end{tikzcd}$$

- *for conscutive morphisms*

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

*we have a commutative diagram*

$$\begin{tikzcd}
\varphi_Z \circ L(fg)^* \arrow[r, "\zeta(fg)"] \arrow[d, "\zeta(g)"'] & \Gamma(fg) \circ \varphi_X \\
\Gamma(g) \circ \varphi_Y \circ Lf^* \arrow[ur, "\Gamma(g)\zeta(f)"'] &
\end{tikzcd}$$

*Proof.* See [14, Theorem 2], where the analogous result is proven with $\Gamma(X) = Inv(X)$ and $\varphi = \det$. The extension to the general case is straightforward, since the original proof of Knudsen and Mumford uses only functorial properties of det functor. □

**Corollary 6.34.** *Let* $\varphi = (\Gamma, \varphi, i)$ *be a Knudsen-Mumford functor and let* $f : X \to Y$ *a perfect morphism of schemes. For each morphism* $g : Y' \to Y$ *consider the base change diagram*

$$\begin{tikzcd}
X \times_Y Y' \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

*We have*

$$\Gamma(g) \circ \varphi_Y \circ Rf_* \simeq \varphi_{Y'} \circ Rp_* \circ Lq^*.$$

*Proof.* Let $\mathcal{F}$ be a perfect complex on $X$. Since $f$ is perfect, $Rf_*(\mathcal{F})$ is perfect on $Y$ and we have natural isomorphisms

$$\Gamma(g) \circ \varphi_Y \circ Rf_*(\mathcal{F}) \simeq \varphi_{Y'} \circ Lg^* \circ Rf_*(\mathcal{F}) \simeq \varphi_{Y'} \circ Rp_* \circ Lq^*(\mathcal{F}).$$

□

Next theorem is the one we need for the applications in our paper. The first part would not require the machinery we have developed in this appendix, but with these tools at our

disposal the proof becomes particularly easy and close to the proof of the most difficult second statement originally due to Knudsen and Mumford.

**Theorem 6.35.** *Let $f : X \to Y$ be a projective morphism of schemes, and let $\mathcal{F}$ be a perfect coherent sheaf on $X$. We have*

(1) $\chi(X_y, \mathcal{F}_y) = \operatorname{rank} Rf_*(\mathcal{F})$ *is independent of* $y \in Y$,
(2) $\mu(X_y, \mathcal{F}_y) \simeq \det Rf_*(\mathcal{F})|_y$ *functorially for all* $y \in Y$.

*Proof.* Since $f$ is projective, it is perfect [2, Exposé III, Proposition 4.8]. Let $g : \{y\} \hookrightarrow Y$ be the inclusion of a point, so that the fiber product $X' = X \times_Y \{y\}$ is the fiber $X_y$ of $f$ over $y$.

To prove (1) consider the Knudsen-Mumford functor rank $= (C(\mathbb{Z}), \mathrm{rank}, \mathrm{id})$, defined by the rank of a locally free sheaves. By corollary 6.34 we get

$$\operatorname{rank} Rf_*(\mathcal{F}) = \operatorname{rank} Rp_*(q^*\mathcal{F}) = \operatorname{rank} Rp_*(\mathcal{F}_y) = \chi(X_y, \mathcal{F}_y).$$

Analogously, if det is the Knudsen-Mumford functor defined by the determinant of locally free sheaves (see example 6.26) we have

$$\det Rf_*(\mathcal{F})|_y = g^* \det Rf_*(\mathcal{F}) \simeq \det Rp_*(q^*\mathcal{F}) = \det Rp_*(\mathcal{F}_y) = \mu(X_y, \mathcal{F}_y),$$

and this proves (2). $\square$

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